Lecture 7 | In order to construct a suitable subalgebra $\mathcal{S}^\pm \subset \mathcal{V}^\pm$ which pairs trivially with $K^\pm = \mathrm{Ker}\,\Upsilon^\pm : \tilde{\mathcal{U}}^\pm \longrightarrow \mathcal{V}^\pm$, we need to recall the pairing

$$\langle \cdot , \cdot \rangle : \mathcal{V}^+ \otimes \tilde{\mathcal{U}}^- \longrightarrow \mathbb{K}$$

$$\langle R^+, f_{i_1,d_1} \dots f_{i_n,d_n} \rangle = \int_{|z_1| \ll \dots \ll |z_n|} \frac{z_1^{d_1} \dots z_n^{d_n} R^+(z_1, \dots, z_n)}{\prod_{1 \le a < b \le n} \zeta_{i_a i_b}\left(\frac{z_a}{z_b}\right)}$$

and seek to rewrite it in terms of

$R^- = \Upsilon^-(f_{i_1,d_1} \dots f_{i_n,d_n})$; naively, it is obvious how to do this: because $\int$ denotes the constant term of a certain rational function, we have $\int$ any function $(z_1, \dots, z_n)$

$$\|$$

$\int$ any function $(z_{\sigma(1)}, \dots, z_{\sigma(n)})$ $\quad \forall\, \sigma \in S_n$

$$\Rightarrow \langle R^+, f_{i_1,d_1} \dots f_{i_n,d_n} \rangle = \frac{1}{n!} \cdot \int R^+(z_1, \dots, z_n) \cdot \mathrm{Sym}\left[\frac{z_1^{d_1} \dots z_n^{d_n}}{\prod_{a<b} \zeta_{i_a i_b}\left(\frac{z_a}{z_b}\right)} \right] = \frac{1}{n!} \int \frac{R^+(z_1, \dots, z_n) R^-(z_1, \dots, z_n)}{\prod_{1 \le a \ne b \le n} \zeta_{i_a i_b}\left(\frac{z_a}{z_b}\right)}$$

However, this can't be right, since it would imply that $\mathcal{V}^+$ pairs trivially with K^- and thus the pairing descends to a pairing $\mathcal{V}^+ \otimes \mathring{\mathcal{S}}^- \longrightarrow \mathbb{K}$ which is non-degenerate in the first argument; since $\mathcal{V}^+ \supsetneq \mathring{\mathcal{S}}^+$ in general, it is quite counter-intuitive to have a non-degenerate pairing between $\mathcal{V}^+$ and a subspace

The error in the logic above is that holds if "any function" is a Laurent polynomial, but not a rational function (or its power series expansion)

Indeed, the correct statement is $\displaystyle\int_{|z_1| \ll \dots \ll |z_n|}$ any function $(z_1, \dots, z_n) = \displaystyle\int_{|z_{\sigma(1)}| \ll \dots \ll |z_{\sigma(n)}|}$ any function $(z_{\sigma(1)}, \dots, z_{\sigma(n)})$

If you wish to reorder the contours of integration of the RHS to $|z_1| \ll \dots \ll |z_n|$ then you will need to account for the residues of the integrand at poles of the form $z_a - z_b \gamma$ as γ runs over $\mathbb{K}^\times$; as we will now see, the corresponding accounting will depend a lot on $\{I, \mathbb{K}, \zeta_{ij}(x)\}$

We will deal with two cases in these lectures: the first is that associated to (preprojective) K-theoretic Hall algebras of quivers corresponding to generic equivariant parameters (see arxiv: 2108.08779)

I = vertex set of a quiver Q

$\mathbb{K} = \mathrm{Frac}(\mathrm{Rep}_T)$, where T is a torus which acts on $T^*\mathrm{Rep}_Q$ by multiplying ϕ_e by a character $t_e : T \to \mathbb{C}^*$, and multiplying ϕ_e^* by a character $\frac{q}{t_e} : T \to \mathbb{C}^*$

We may interpret $t_e, \frac{q}{t_e} \in \mathrm{Rep}_T$ as lying in $\mathbb{K}$, and we assume that they satisfy the following genericity condition: $\exists$ one-parameter subgroup $\mathbb{C}^* \subset T$ with respect to which $t_e|_{\mathbb{C}^*}$ sends $z \to z^*$ and $\frac{q}{t_e}|_{\mathbb{C}^*}$ sends $z \to z^\#$ for $*, \# > 0$.

$$\varphi_{ij}(x) = \left[\frac{(1-xq^{-1})}{(1-x)}\right]^{\delta_{ij}} \prod_{e = \vec{ij}} \left(\frac{1}{t_e} - x\right) \prod_{e = \vec{ji}} \left(1 - \frac{t_e}{qx}\right)$$

note that each monomial corresponds to one of ϕ_e or ϕ_e^*; the overall twist (a power of x times $\pm$ a character) was chosen so that $\exists$ well-defined double

The assumption on $t_e, \frac{q}{t_e}$ above will allow us to work with $t_e, \frac{q}{t_e}$ as if they were complex numbers of absolute value < 1; this will be handy in residue computations

$$\langle R^+, f_{i_1,d_1} \cdots f_{i_n,d_n} \rangle = \int_{|z_1| \ll \cdots \ll |z_n|} \frac{R^+(z_1, \ldots, z_n)\, z_1^{d_1} \cdots z_n^{d_n}}{\prod_{a<b} \varphi_{i_a i_b}\left(\frac{z_a}{z_b}\right)} \qquad (*)$$

Let us try to move the contours toward $|z_1| = \cdots = |z_n|$; the reason for this is that the latter contours are invariant with respect to repermuting the variables, and so it is true that $(*)$ above equals $\frac{1}{n!} \int_{|z_1| = \cdots = |z_n|} \frac{R^+(z_1, \ldots, z_n) R^-(z_1, \ldots, z_n)}{\prod_{1 \le a \neq b \le n} \varphi_{i_a i_b}\left(\frac{z_a}{z_b}\right)}$

What poles do we encounter as we move the variables from $|z_a| \ll |z_b|$ to $|z_a| = |z_b|$?

If we assume that $|t_e|, \frac{|q|}{|t_e|} < 1$, then the linear factors $\frac{1}{t_e} - \frac{z_a}{z_b}$ and $1 - \frac{t_e z_b}{q z_a}$ do not produce poles; however, note that the linear factor $\left(1 - \frac{z_a}{z_b q}\right)^{\delta_{i_a i_b}}$ does produce poles and so as we move contours from $|z_1| \ll \cdots \ll |z_n|$ we will pick up poles of the form

$$z_{a_1} = q z_{a_2} = \cdots = q^{s-1} z_{a_s}$$

for various $a_1 < a_2 < \cdots < a_s$ with $i_{a_1} = \cdots = i_{a_s}$; but now assume we have the residue at $z_{a_1} \overset{q}{—} z_{a_2} \overset{q}{—} \cdots \overset{q}{—} z_{a_s}$ and z_b much larger for some $b > a$

What kind of pole can we pick up as z_b passes over the geometric progression $q^{s-1} z_{a_s}, \dots, q z_{a_s}, z_{a_s}$? Because of the linear factors in ζ_{ij} there are three types of residues which may appear:

- $z_b = t_e q^x z_{a_s}$ $\forall$ edge $e : i_{a_s} \to i_b$ and $\forall\, x \in \{0, \dots, s-1\}$
- $z_b = \dfrac{q^{x+1}}{t_e} z_{a_s}$ $\forall$ edge $e : i_b \to i_{a_s}$ and $\forall\, x \in \{0, \dots, s-1\}$
- $z_b = \dfrac{z_{a_s}}{q}$ if $i_{a_s} = i_b$

The last pole simply has the effect of adding one element to the geometric progression; if we can get rid of the first poles for all $x < s-1$, then we will be able to move the contour of $|z_b|$ all the way to the contour of $|z_{a_1}|$ and only be hindered by the last pole; in order to achieve this, we define

$$\mathcal{S}^+ = \left\{ R(z_{ia}) \;\middle|\; \forall\, i,j \in I,\ \ \prod_{e: \vec{ij}} \left(z_{ia} - \frac{q z_{j1}}{t_e}\right) \prod_{e = \vec{ji}} \left(z_{ia} - t_e z_{j1}\right) \Big|\ R\big|_{z_{j2} = q z_{j1}} \atop \forall\, (i,a) \notin \{(j,1),(j,2)\} \right\}$$

which is a subspace of $\mathcal{V}^+$ a priori; the condition above essentially requires that for any $\gamma \in \mathbb{K}^\times$, we have $(z_{ia} - \gamma z_{j1})^{\#}$ divides $R\big|_{z_{j2} = q z_{j1}}$, where $\#$ is the number of arrows $\vec{ij}$ s.t. $\frac{q}{t_e} = \gamma$ $+$ the number of arrows $\vec{ji}$ s.t. $t_e = \gamma$

The residue discussion above implies the following:

$\boxed{\text{Prop}}$: for any $R^+ \in \mathcal{S}^+$, and any $i_1, \dots, i_n \in I$, $d_1, \dots, d_n \in \mathbb{Z}$, we have

$$\langle R^+, f_{i_1,d_1}, \dots, f_{i_n,d_n} \rangle = \sum_{\substack{\text{partitions of } \{1,\dots,n\} \\ \text{into } \{a_1 < \dots < a_s\} \text{ s.t. } i_{a_1} = \dots = i_{a_s} \\ \{b_1 < \dots < b_t\} \text{ s.t. } i_{b_1} = \dots = i_{b_t} \text{ etc}}} \int_{|z_{a_1}| = |z_{b_1}| = \dots} \operatorname{Res} \frac{R^+(z_1, \dots, z_n)\, R^-(z_1, \dots, z_n)}{\prod_{1 \le a \ne b \le n} \zeta_{i_a i_b}\left(\frac{z_a}{z_b}\right)}$$

where $R^- = \Upsilon^-(f_{i_1,d_1} \dots f_{i_n,d_n})$ and Res denotes the residue at $z_{a_2} = z_{a_1}/q, \dots, z_{a_s} = z_{a_1}/q^{s-1}$

$\Downarrow$

We get an induced pairing

$$\mathcal{S}^+ \otimes \mathring{\mathcal{S}}^- \xrightarrow{\langle\cdot,\cdot\rangle} \mathbb{K}.$$

Prop: $\mathcal{S}^+$ is an algebra, i.e. closed under the shuffle product.

Proof: take any $R_1, R_2 \in \mathcal{S}^+$ and any $\gamma \in \mathbb{K}^\times$; for any $i,j \in I$ we have

$$R_1 * R_2 = \mathrm{Sym}\left[\frac{R_1(z_{ia},\ldots,z_{jb},\ldots)}{n_1!} \frac{R_2(z_{ia'},\ldots,z_{jb'},\ldots)}{n_2!} \prod_{a,b'} \zeta_{ij}\left(\frac{z_{ia}}{z_{jb'}}\right) \prod_{a',b} \zeta_{ji}\left(\frac{z_{jb}}{z_{ia'}}\right) \cdots \right]$$

now let's set $z_{j2} = q z_{j1}$ in $R_1 * R_2$; in each term in the RHS we have two options

- z_{j1} and z_{j2} are among the variables of R_1 (respectively R_2); then if
  - z_{ia} is also among the variables of R_1 (respectively R_2), then the fact that R_1 (resp. R_2) $\in \mathcal{S}^+$ ensures the fact that $(z_{ia} - \gamma z_{j1})^\# \mid R_1 * R_2|_{z_{j2} = z_{j1}q}$
  - z_{ia} is among the variables of R_2 (respectively R_1); then the linear factors of the form $\zeta_{ji}\left(\frac{z_{j1}}{z_{ia}}\right)$ (resp. $\zeta_{ij}\left(\frac{z_{ia}}{z_{j2}}\right)$) ensure $(z_{ia} - \gamma z_{j1})^\#$ divides $R_1 * R_2|_{z_{j2} = z_{j1}q}$
- z_{j1} and z_{j2} are each among the variables of one of R_1 and R_2 then because of the linear factors in $\zeta_{jj}\left(\frac{z_{j1}}{z_{j2}}\right)$ or $\zeta_{jj}\left(\frac{z_{j2}}{z_{j1}}\right)$, the only way the evaluation at $z_{j2} = q z_{j1}$ can be non-zero is if z_{j1} is a variable of R_1 and z_{j2} is a variable of R_2; in this case, the linear factors of the form $\zeta_{ji}\left(\frac{z_{j1}}{z_{ia}}\right)$ (if z_{ia} is among the variables of R_2) or the linear factors of the form $\zeta_{ij}\left(\frac{z_{ia}}{z_{j2}}\right)$ (if z_{ia} is among the variables of R_1) ensure $(z_{ia} - \gamma z_{j1})^\#$ divides $R_1 * R_2|_{z_{j2} = z_{j1}q}$

Corollary: $\mathring{\mathcal{S}}^+ \subseteq \mathcal{S}^+$ (by the way, $\mathcal{S}^- \subset \mathcal{V}^-$ is defined likewise)

$\Downarrow$ with some work due to the infinite-dimensionality of the spaces involved

the non-degeneracy of the pairing $\mathcal{S}^+ \otimes \mathring{\mathcal{S}}^- \to \mathbb{K}$ in the first argument implies

$\mathcal{S}^+ = \mathring{\mathcal{S}}^+$

and so the pairing $\mathcal{S}^+ \otimes \mathcal{S}^- \longrightarrow \mathbb{K}$ is non-degenerate in both arguments

$$\begin{array}{ccc} \mathcal{S}^+ & \otimes & \mathcal{S}^- \\ \| & & \| \\ \mathring{\mathcal{S}}^+ & \otimes & \mathring{\mathcal{S}}^- \\ \wr\| & & \wr\| \\ U^+ & \otimes & U^- \end{array}$$

Next time we will use the fact that $\mathcal{S}^\pm \subset \mathcal{V}^\pm$ is cut out by explicit linear conditions to obtain (by dualizing) generators for the kernel of $\tilde{U}^\mp \twoheadrightarrow U^\mp$.

The fact that $\mathcal{S}^+ = \mathring{\mathcal{S}}^+$, proved on the previous page, has the following important consequence pertaining to K-theoretic Hall algebras.

<u>Corollary</u>: let us take the (analogue in the situation at hand) of the morphism

$$K\text{-}HA_{T,Q,loc} \xrightarrow{i_*} \mathcal{V}^+$$

of Lecture 4 (the subscript T means that we work equivariantly w.r.t the chosen characters $t_e, \frac{q}{t_e} : T \to \mathbb{C}^*$, while the subscript loc means we tensor with $\mathbb{K} = \mathrm{Frac}(\mathrm{Rep}_T)$ over Rep_T stands for "localization"); then

$$\mathrm{Im}\ i_* = \mathcal{S}^+$$

Proof: the fact that $\mathrm{Im}\ i_* \supseteq \mathring{\mathcal{S}}^+$ follows from the fact that $\mathcal{S}^+ = \mathring{\mathcal{S}}^+$ and all its generators z_{i1}^d lie in $\mathrm{Im}\ i_*$ (see page 11). It remains to prove that $\mathrm{Im}\ i_* \subseteq \mathcal{S}^+$, which we will do by adapting a classic argument by Yu Zhao. Pick any $i, j \in I$, $\gamma \in \mathbb{K}^*$ and let $\#$ be the non-negative integer on page 21. We need to show that for any $\Gamma \in K_T(T^*\mathrm{Rep}_Q)$, the class

$$F = i_*(\Gamma) \in K_{T\times G}(\text{affine space}) = \mathbb{K}[z_{j1}^{\pm 1}, z_{j2}^{\pm 1}, \dots, z_{ia}^{\pm 1}, \dots]^{sym}$$

has the property that $F|_{z_{j2} = q z_{j1}}$ is divisible by $(z_{ia} - \gamma z_{j1})^{\#}$.

To this end, consider the set of edges $e_1, \dots, e_d : \vec{ji}$ s.t. $t_{e_1} = \dots = t_{e_d} = \gamma$

$e'_1, \dots, e'_{\#-d} : \vec{ij}$ s.t. $\frac{q}{t_{e'_1}} = \dots = \frac{q}{t_{e'_{\#-d}}} = \gamma$

consider the locally closed subspace $U \subset \left(\text{affine space of } \phi_e : \mathbb{C}^{n_j} \rightleftarrows \mathbb{C}^{n_i} : \phi_e^*\right)_{e \in E}$ consisting of linear transformations (ϕ_e, ϕ_e^*) whose only non-zero matrix coefficients are $(\phi_{e_r})_{a1} = x_r$, $(\phi_{e_r}^*)_{2a} = y_r$, $\forall r \in \{1, \dots, d\}$

$$(\phi_{e'_{r'}})_{2a} = y'_{r'}, \quad (\phi^*_{e'_{r'}})_{a1} = x'_{r'}, \quad \forall r' \in \{1, \dots, \#-d\}$$

with respect to a given basis of $\mathbb{C}^{n_i}$ and $\mathbb{C}^{n_j}$, where $x_1, \dots, x_d, y_1, \dots, y_d, x'_1, \dots, x'_{\#-d}, y'_1, \dots, y'_{\#-d}$ are complex numbers which satisfy $x_1 y_1 + \dots + x_d y_d \neq x'_1 y'_1 + \dots + x'_{\#-d} y'_{\#-d}$.

The point of the inequality above is that $T^*\mathrm{Rep}_Q \cap U = \emptyset$ in the affine space of ϕ_e, ϕ_e^*

$\Rightarrow F|_U = 0$; however, U is an affine bundle over projective space, whose

K-theory corresponds to $\mathbb{K}[z_{j1}^{\pm 1}, z_{j2}^{\pm 1}, \dots, z_{ia}^{\pm 1}, \dots] \big|_{z_{j2} = q z_{j1}} / (z_{ia} - z_{j1}\gamma)^{\#}$; see Prop 5.10 in 2108.08779